ADVANCES IN MATHEMATICS 19, 207–237 (1976)

Strict, Weighted, and Mixed Topologies and Applications*

H. S. COLLINS

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana 70803

Communicated by G.-C. Rota

DEDICATED TO THE MEMORY OF PASQUALE PORCELLI

INTRODUCTION

The purpose of this paper is to give a brief survey of certain special but rather important topologies that occur in functional analysis, topologies that have applications in approximation theory, analytic function theory, the spectral analysis of bounded continuous functions, probability and topological measure theory, interpolation theory, and certain aspects of C^*-algebra theory. An attempt is made here to define "strict," "weighted," and "mixed" topologies in a reasonably general way, so that most of the interesting theory and examples are included. The meaning attached here, particularly for strict, may be more inclusive than that traditionally encountered in the rather extensive literature. For example, most of the important weighted and mixed topologies, and certain Hausdorff inductive limit topologies that have arisen lately, as well as the traditional strict topology, will be strict topologies (in our sense). To balance this somewhat broad interpretation of strict topology, we restrict ourselves to three basic situations where the underlying vector space is either (1) a space of continuous real or complex valued functions on a completely regular Hausdorff space X, (2) the double centralizer algebra $M(A)$ of a C^*-algebra A, or (3) the double centralizer algebra $\Gamma(K)$ of Pedersen's ideal K of a C^*-algebra A. This means we include very little discussion of a fairly large body of work on strict topologies on Banach modules, although a number of references are made to them here. Some attempt will be made to provide both historical motivation for these topologies and a comprehensive exposition of some of the more interesting applications. In fact, applications will be emphasized since no really cohesive theory has yet arisen.

* This research was supported in part by NSF. Grant GP-20866-A No. 1.

An obvious point of departure (at least from our point of view) is the Riesz–Markoff theorem: The Banach space dual of $C_0(S)$ (= the supremum normed commutative C^*-algebra of all complex valued continuous functions that vanish at infinity on the locally compact Hausdorff space S) is $M_b(S)$, the variation normed Banach space of bounded Radon measures on S. We are thus considering the pairing $\langle f, \mu \rangle$, where $f \in C_0(S)$, $\mu \in M_b(S)$, and $\langle f, \mu \rangle = \int f \, d\mu$ (here and throughout, references for topological vector space concepts are as in the standard references, e.g., [74]). A question of some interest, first examined by Buck in [11, 13] is the following: Does there exist a complex vector space V containing $C_0(S)$ and a Hausdorff locally convex topology (= LCS *topology*) β on V so that the topological dual $(V, \beta)'$ of (V, β) is $M_b(S)$? He showed V could be taken to be $C_b(S)$, all bounded continuous complex valued functions on S, and β to be the topology for which $f \to \| \varphi f \|_\infty$, $f \in C_b(S)$, $\varphi \in C_0(S)$ gives a defining family of seminorms ($\| \ \|_\infty$ = supremum norm). Buck called β the strict topology and besides making a reasonably detailed study of the topological algebra $(C_b(S), \beta)$, raised three especially interesting questions (actually question (2) was first posed in [18, p. 167]): (1) What are the bounded sets and convergent sequences of $(C_b(S), \beta)$? (2) If β' = the finest LCS topology on $C_b(S)$ agreeing with β on β (or norm) bounded sets, is $\beta' = \beta$? (3) Is β the finest LCS topology on $C_b(S)$ having $M_b(S)$ as dual? The last question may be phrased: "*Is* $(C_b(S), \beta)$ *a Mackey space*?" These and allied problems (Buck earlier in [12] had suggested possible uses of β in analytic function theory and Herz in [45] has rephrased some earlier work of Beurling [5] on spectral analysis in terms of β) have generated considerable activity since 1960 and have led to a study of generalizations of these topologies to Banach modules, strict topologies on double centralizer algebras, weighted and strict topologies on spaces of continuous functions, and a renewed interest in the mixed topologies of Wiweger [107]. Ironically, only a few results appear in the literature that deal with the applications of strict topologies in Harmonic analysis (e.g., see the first part of Section 4 below), although such results seem to have been the earliest that make use of these topologies. On the other hand, extensive use has been made and is now being made of both strict and weighted topologies in the other areas of application: approximation theory, analytic function theory, probability and topological measure theory, interpolation theory, and applications to a study of C^*-algebras. It is to these aspects of the theory that we address ourselves primarily. First, however, we give some preliminary discussion (Section 1) and our rather general definitions (Section 2) of strict topologies. In the second section, we also define weighted and mixed topologies.

1. Preliminaries

Throughout this paper, S is a locally compact Hausdorff space, X is a completely regular Hausdorff space, $C_0(S)$ is the commutative C^*-algebra of all continuous complex valued functions on S that vanish at infinity, $C_b(X)$ is the bounded continuous scalar valued (real or complex) functions on X, A is not necessarily commutative C^*-algebra, $M(A)$ is (resp. $\Gamma(K)$) the double centralizer algebra of A (resp. of Pedersen's ideal K of A), and $M_b(S)$ is the Banach space of bounded Radon measures on S. In Section 3, we introduce and examine several significant classes of Baire (and Borel) measures on X that arise naturally, namely, the tight, τ-additive, separable, and σ-additive measures, denoted (respectively) by M_t, M_τ, M_S or M^∞, and M_σ. Of course, it is understood that for each $\eta = t, \tau, S$ or σ, $M_\eta = M_\eta(X)$.

If E is an LCS (locally convex T_2 topological vector space), E' will denote its dual space of continuous linear functionals, and if a different LCS topology t occurs on E, $(E, t)'$ will be used. For example, the *Mackey problem for* E is the following: Is the given topology on E the finest LCS topology t on E such that $E' = (E, t)'$? Another problem of interest that arises is in determining when a space X is Prohorov, i.e., when a weak* compact set of $(E, t)'$ consisting of positive measures is equicontinuous on (E, t), where $E = C_b(X)$ and t is the substrict topology β_0 on E. The connection between these two problems is perhaps best seen by recalling the well-known characterization of Mackey spaces: (E, t) is a Mackey space iff each weak* compact absolutely convex subset of $(E, t)'$ is t equicontinuous. Another related problem is to find conditions that ensure (E, t) is a *strong Mackey space*, i.e., each relatively weak* countably compact subset of $(E, t)'$ is t equicontinuous.

2. Definition of Strict, Mixed, and Weighted Topologies

As mentioned above, the first use of the word strict with reference to a topology was by Buck in [11, 13], although Herz in [45] also used the term and indicated that both Beurling and Domar were familiar with these or like topologies. All these mathematicians first defined the strict topology as a weighted topology, although the word was not yet in use (see Introduction for Buck's definition). Herz [45] gave an alternative description that in virtue of [107, Theorem 3.1.1] showed that β on $C_b(S)$ is a mixed topology. We now give our versions of these three types of topologies. The reader will note that these definitions are not (in the case of strict and mixed) the same as those originally given: The strict definition is more general and the mixed one slightly less general.

2.1. Definition of Strict Topology. Let E be an LCS with topology t and denote by t' the finest LCS topology on E agreeing with t on t bounded sets (such a topology always exists and has a base at zero consisting of all absolutely convex sets W so that B a t bounded set implies that there is an absolutely convex t neighborhood V $(= V_{W,B})$ of zero so that $V \cap B \subseteq W \cap B$). This LCS topology t' is *the strict topology* of (E, t) or *the strict topology on E generated by t*.

An early and fairly comprehensive study of such topologies appears in [16], where, for example, the Grothendieck completion of a barrelled [74] LCS, E, is simply $(E', t')'$, where t is the weak* topology of E'. It is also shown in [39, Theorem 3] that the topology on any (DF) space is the strict topology.

2.2. Definition of Weighted Topology. Here, the space $E = E_V$ is a subspace of the continuous scalar valued functions on X and is determined at the same time the weighted topology ω_V is determined. We have a *Nachbin family* V of functions on X (the term is Summers', [89, 90]), i.e., a family V of nonnegative upper-semicontinuous (u.s.c.) functions on X such that u, $v \in V$ and $\lambda \geqslant 0$ implies there is $w \in V$ so that $\lambda u \leqslant w$ and $\lambda v \leqslant w$ (pointwise). The weighted space E_V and weighted topology ω_V are then defined as follows: $E_V =$ all continuous scalar valued functions f on X such that fv is bounded and vanishes at infinity for every $v \in V$ (*fv vanishes at infinity* means $\{x \in X: |f(x)|\, v(x) \geqslant \epsilon\}$ is relatively compact for all $\epsilon > 0$), and ω_V is that topology on E_V whose seminorms are given by $f \to \|fv\|_\infty$, $v \in V$, $f \in E_V$, where $\|fv\|_\infty =$ supremum norm of fv.

In Sections 2.4 and 5, we mention extensions of 2.2 to non-Abelian settings.

2.3. Definition of Mixed Topology. As remarked above, our definition here is somewhat less general than Wiweger's [107] and in fact, is exactly that given by Cooper in [25]. Let (E, t) be an LCS, and let t^* be another LCS topology on E so that $t^* \leqslant t$ (hence, t bounded sets are t^* bounded). Suppose also that (E, t) is a (DF) space [39] with a base for t bounded sets consisting of a *sequence* $\{B_n\}$ of absolutely convex t bounded sets B_n so that $B_n + B_n \subseteq B_{n+1}$ for all n and each B_n is t^* closed. Under these conditions, the *mixed topology* $m = m(E, t^*, t)$ on E is that LCS topology whose base at zero consists of all sets of the form $\bigcup_{n=1}^{\infty} [(U_1^* \cap B_1) + (U_2^* \cap B_2) + \cdots + (U_n^* \cap B_n)]$, where $\{U_n^*\}$ is a sequence of absolutely convex t^* neighborhoods of zero.

2.4. CONNECTIONS BETWEEN THESE TOPOLOGIES. In [25, Proposition 1], Cooper shows that each mixed topology $m = m(E, t^*, t)$ is the strict topology of (E, m), so that *every mixed topology is a strict topology.* The question (is $\beta = \beta'$ on $C_b(S)$?) mentioned in the Introduction, has been answered affirmatively and independently by several different mathematicians (see [26], [55, Proposition 2] in conjunction with [23, Theorem 2.2], [107, 2.2.2 and 3.1.1], [25, Propositions 1 and 3]), so this says *the topology β of Buck is the strict topology of* $(C_b(S), \beta)$, and is the finest LCS topology on $C_b(S)$ agreeing with the compact-open topology on norm balls. By its very definition (and the fact [13, Lemma 4] that if $V = C_0(S)^+ =$ all nonnegative functions in $C_0(S)$, then $E_V = C_b(S)$), the original β of Buck is a weighted topology. As mentioned earlier, Herz [45], remarked that β is also a mixed topology (and Cooper shows this also in [25, Proposition 3]), so β is a strict, weighted, and mixed topology.

Recent developments have focused attention on three main generalizations of Buck's topology, namely, (1) in generalizations to the double centralizer algebra $M(A)$ of a C^*-algebra A or to the double centralizer algebra $\Gamma(K)$ of Pedersen's ideal K of A (see [15, 31, 32, 53, 54, 57, 98–100]), (2) in general weighted spaces and weighted topologies in function spaces a la 2.2 (see [6–8, 65–67, 70, 89–97], and (3) in various generalizations of Buck's topology to $C_b(X)$, the bounded continuous scalar valued functions on a completely regular (not necessarily locally compact) Hausdorff space (see [25, 33, 35, 40, 41, 44, 46, 59, 63, 64, 82, 84, 92, 97, 102, 105–107]).

In (1), we regard $(M(A), \beta)$ as the non-Abelian version of $(C_b(S), \beta)$ and $(\Gamma(K), \kappa)$ as the non-Abelian analog of ($C(S)$, compact-open): $C(S) =$ *all* continuous scalar valued functions on S. Here, Taylor in [98, p. 638] shows that β on $M(A)$, defined initially as a weighted topology given by the seminorms $x \to \max(\| ax \|, \| xa \|)$, $x \in M(A)$, $a \in A^+$, is the strict topology of $(M(A), \beta)$. It also follows from his results easily that $\beta = m(M(A), t^*, t)$, where t is the norm topology of $M(A)$ and t^* is the topology on $M(A)$ defined by any fixed bounded approximate identity $\{e_\lambda : \lambda \in \Lambda\}$ for A according to the formula: $x_\alpha \to x(t^*)$ iff $\| e_\lambda(x_\alpha - x)\| \to_\alpha 0$ and $\|(x_\alpha - x) e_\lambda \| \to_\alpha 0$, for all $\lambda \in \Lambda$ (in case $A = C_0(S)$ so that $M(A) = C_b(S)$, then t^* would be the compact-open topology of $C_b(S)$). In (2), most of the interesting locally convex spaces of continuous functions occur, and have their best applications to approximation theory, expecially to the weighted approximation problem [66, 93, 94]. So far as we know, virtually nothing has been done regarding the general problem as to when a weighted topology is a strict or mixed one, although it frequently is (see, for example, the theorem later in this section). One result along these lines is that of Prolla [70], which says that *any* LCS is topologically isomorphic with a subspace of some weighted space, a result that unhappily seems of little use. In (3), early work in probability and topological measure theory [55, 103] led Sentilles [82] and Sentilles–Wheeler [84] to define four topologies $\xi = \beta_0$, β, β_e and β_1 on $C_b(X)$ so that $(C_b(X)\ \xi)'$ yields four important and widely studied spaces of Baire measures, namely, M_t, M_τ, M_S, and M_σ (see [3, 4, 28, 33, 35, 38, 42–44, 46, 49–51, 55, 56, 60–64, 72, 73, 92, 102, 103, 105, 106]; more will be said about these and allied spaces in Section 3 below). The substrict topology β_0 is strict, mixed, and weighted [25, 35, 46, 92, 102], exactly as in Buck's case, where X is locally compact. The most general work here seems to be that of Mosiman, where he proves [63, Proposition 2.4] each of his topologies β_L on $C_b(X)$ satisfies $\beta_0 \leqslant \beta \leqslant \beta_L \leqslant$ norm and each β_L is a strict topology in our sense. Further, [63, Theorem 6.1] each of β, β_e, and β_1 is β_L for appropriate L (here L is a collection of compact subsets of $\beta X\backslash X$ and β_L is defined as the inductive limit [74] of the spaces $[(C_b(\beta X\backslash Q), \beta_Q), T_Q]$, where $\beta X =$ the Stone–Čech compactification of X, $Q \in L$, β_Q is Buck's topology on $C_b(\beta X\backslash Q)$, and $T_Q : C_b(\beta X\backslash Q) \to C_b(X)$ is restriction. This procedure generalizes the method introduced by Sentilles in [82] to obtain β and β_1). In this setting, we record the following simple theorem: *$(C_b(X), \beta_L)$ is a weighted space iff $\beta_L = \beta_0$.* Since one way is clear from the preceding remark, it suffices to show that if $(C_b(X), \beta_L) = (E_U, \omega_U)$ for a Nachbin family U on X, then $\beta_L = \beta_0$. Now, $1 \in C_b(X) = E_U$, so $u \in U$ implies $\{x \in X: u(x) \geqslant \epsilon\}$ is relatively compact for each $\epsilon > 0$, hence, $U \subseteq V =$ the Nachbin family of all nonnegative upper semicontinuous functions that vanish at infinity. By [89, Theorem 3.1] (with $E = C_b(X)$), $E_V \subseteq E_U = E$ and the restriction of $\beta_L = \omega_U$ to E_V is $\leqslant \omega_V$. But, by [92, p. 192], $E = E_V$ and $\omega_V = \beta_0$, so $\beta_0 \leqslant \beta_L \leqslant$

$\omega_V = \beta_0$, and $\beta_L = \beta_0$. Thus, it follows that (the always strict topology) *β_L is weighted implies β_L is also mixed* (since β_0 is). However, nothing seems to be known in general about when the topologies β_L are mixed. Of course, this is true when $\beta_L = \beta_0$ and it is this fact (more accurately, the theorem above) that we use below in our discussion of applications of strict topologies to topological measure theory.

3. Applications to Topological Measure Theory

The papers of Le Cam [55] and Varadarajan [103] are especially important to us here from both a historical, motivational, and mathematical viewpoint, and serve as the point of departure for this section. Le Cam introduced [55, p. 212] the spaces of tight, τ-additive, and σ-additive functionals on $C_b(X)$, and made an extensive study of weak* compactness of subsets and weak* convergence of sequences in these spaces of functionals. Varadarajan [103] obtained integral representations of these functionals in terms of tight, τ-additive, and σ-additive (Baire) measures (see also [82, p. 313–314]) and continued and extended some of these investigations, as did Knowles [51], Moran [60–62], Granirer [38], Dudley [28], Kirk [49, 50], Leger and Soury [56], Rome [72, 73], Haydon [42–44], Buchwalter [14], and Berruyer–Ivol [3, 4]. Le Cam [55, Proposition 2] introduced a topology T_t on $C_b(X)$ that gave M_t as dual: The topology of uniform convergence on the norm-bounded and tight (or uniformly tight) subsets of M_t, where H is *uniformly tight* if $\epsilon > 0$ implies there is compact K_ϵ so that $|\mu|(X \backslash K_\epsilon) < \epsilon$ for all $\mu \in H$. Here, we regard the μ's as Borel measures (see [33]). Le Cam [55, Proposition 2] showed that T_t is the finest LCS topology agreeing with the compact-open topology on norm balls, and showed [55, proposition 3] that when X is locally compact and σ-compact, then $(C_b(X), T_t)$ is a strong Mackey space. Conway independently in [23, Theorem 2.2], again for locally compact X, described the β equicontinuous subsets of M_t as the norm-bounded uniformly tight ones. This, together with [55, Proposition 2] yields $\beta' = \beta$. Conway also showed $(C_b(X), \beta)$ is a strong Mackey space when X is locally compact paracompact [23, Theorem 2.6], and in [24], obtained some quite general results on convergence of sequences of tight measures, including as a corollary an interpolation theorem of Bade [2].

The early work of Buck, Le Cam, and Varadarajan led somewhat later to the introduction by Sentilles in [82] and Fremlin–Garling–Haydon independently in [33] of the topologies β_0, β, and β_1 (called T_t, T_τ, and T_σ by the latter) on $C_b(X)$ that yield as dual spaces the spaces of measures M_t, M_τ, and M_σ, respectively (in between, some partial results along these lines were obtained by Van Rooij [102], Summers [92], and Hoffman–Jørgensen [46]).

At this point, because of the importance of these topologies, both as a unifying force and as a tool in the study of topological measure theory, we pause to formally state some of the major results. We also include some examples and proofs of results (here and in later sections). At several places, the "l^∞ trick" (sometimes called the "method of the gliding or sliding hump") is used, e.g., see the proofs below of Theorems 3.5(b), 3.6(c), and 5.6(c, i). First, we give the definitions of β_0, β, β_1, and β_e.

DEFINITION 3.1. The topology β_0 on $C_b(X)$ is the finest LCS topology agreeing with κ, the compact-open topology, on norm-bounded sets. The topology β (resp. β_1) is the topology β_L on $C_b(X)$ (see 2.4 above), where $L =$ *all* compact subsets of $\beta X \backslash X$ (resp. $L =$ all zero sets of $\beta X \backslash X$). Finally, β_e is the finest LCS topology on $C_b(X)$ agreeing with κ on the uniformly bounded and equicontinuous subsets of $C_b(X)$.

DEFINITION 3.2. Following Moran [60–62], we say X is *measure compact* (resp. *strongly measure compact*) if $M_\sigma = M_\tau$ (resp. $M_\sigma = M_t$). After Mosiman–Wheeler in [64], we say X is β *simple* if $\beta_0 = \beta_1$.

Remark 3.3. We shall use freely in the sequel the various versions of β_0 (which were mentioned in 2.4 above). For example, β_0 is given by the families $\mathscr{P}$ and $\mathscr{Q}$ of seminorms [46, 92], where $q \in \mathscr{Q}$ iff there is a strictly increasing sequence $\{a_n\}$ of positive scalars and an increasing sequence $\{K_n\}$ of compact sets so that

$$q(f) = \sup\{(1/a_n) \|f\|_{K_n} : n = 1, 2, 3,\ldots\}$$

and $p \in \mathscr{P}$ iff there is a nonnegative v so that $\{x \in X: v(x) \geq \epsilon\}$ is compact for each $\epsilon > 0$ and $p(f) = \|fv\|_\infty$, all $f \in C_b(X)$. Here, of course, $\|f\|_K = \sup\{|f(x)| : x \in K\}$.

In the remainder of this section, X will be a completely regular Hausdorff space, unless otherwise specified.

THEOREM 3.4. *Let X and β_0, β, β_1, and β_e be defined as above. Then*:

(a) $(C_b(X), \beta_0)' = M_t$, $(C_b(X), \beta)' = M_\tau$, $(C_b(X), \beta_1)' = M_\sigma$, and $(C_b(X), \beta_e)' = M_S$.

(b) *All of* β_0, β, β_1, *and* β_e *are strict topologies in our sense* (2.1 *above*), *and the* ξ *bounded subsets of* $C_b(X)$ *are exactly the norm-bounded sets,* $\xi = \beta_0$, β, *and* β_1.

(c) $(C_b(X), \xi)$ *is a weighted space iff* $\xi = \beta_0$ ($\xi = \beta$ *or* β_1).

(d) *Since* $\beta_0 \leqslant \beta \leqslant \beta_e \leqslant \beta_1$, *always* $M_t \subseteq M_\tau \subseteq M_S \subseteq M_\sigma$, *with the inequalities generally strict.*

(e) *The following are equivalent*: (i) $(C_b(X), \beta_0)$ *is complete,* (ii) $(C_b(X), \beta_0)$ *is quasi-complete,* (iii) X *is a* k'*-space, i.e., each scalar valued function on* X *whose restriction to each compact set is continuous is also continuous.*

Proof. (a) We give here the proof only for β_0 (the remainder may be found in [33, 82, 84, 105]. If $\mu \in M_t$, let $F(f) = \int f\, d\mu$, $f \in C_b(X)$. There exists an increasing sequence $\{K_n\}$ of compacta such that $|\mu|(X \backslash K_n) < 2^{-2n}$. Let $q(f) = \sup_n\{(1/2^{n-2}) \|f\|_{K_n}\}$, so that $q(f) \leqslant 1$ implies $|F(f)| \leqslant |\mu|(X) + 1$; i.e., F is β_0 continuous by Remark 3.3 above. For the converse, if F is β_0 continuous, it is also norm continuous, so by the Riesz–Markov theorem, there exists $\nu \in M_b(\beta X)$ so that $F(f) = \int f'\, d\nu$, $f \in C_b(X)$, where f' is the unique extension of f to be continuous over βX. By hypothesis, there exists a strictly increasing sequence $\{a_n\}$ of positive numbers and an increasing sequence $\{K_n\}$ of compacta so that $|F(f) \leqslant 1$ for every f such that $q(f) = \sup_n\{(1/a_n \|f\|_{K_n}\} \leqslant 1$. If f is such, $\int |f'|\, d|\nu| = \sup\{|\int f'g'\, d\nu| : g \in C_b(X), \|g\|_\infty \leqslant 1\} = \sup\{|F(fg)| : g \in C_b(X), \|g\|_\infty \leqslant 1\}$. But note that $\|g\|_\infty \leqslant 1$ implies $q(fg) \leqslant 1$, so $|F(fg)| \leqslant 1$, i.e., we have shown $\int |f'|\, d|\nu| \leqslant 1$ if $q(f) \leqslant 1$. There exist open sets U_n in βX containing K_n such that $|\nu|(U_n \backslash K_n) < (1/a_n)$ and $f_n \in C_b(X)$ such that image $f_n' \subseteq [0, a_n]$, $f_n' = 0$ on K_n, $f_n' = a_n$ in $\beta X \backslash U_n$. Then, $\|f_n\|_{K_j} = 0 \leqslant a_j$ if $1 \leqslant j \leqslant n$ and $\|f_n\|_{K_j} \leqslant a_n \leqslant a_j$, $j \geqslant n$. Thus, $q(f_n) \leqslant 1$ and $a_n |\nu|(\beta X \backslash K_n) \leqslant 2$ for all n, hence, $|\nu|(\bigcup_{n=1}^\infty K_n) = |\nu|(\beta X)$. Define μ to be ν restricted to X. We then have $F(f) = \int f'\, d\nu = \int f\, d\mu$, since ν vanishes outside $\bigcup_{n=1}^\infty K_n$, and this completes the proof of (a).

For (b), see [63, 82], and note that the proof of (c) is given at the end of 2.4 above for general β_L.

(d) The example of [33, p. 125] shows $M_t \subsetneqq M_\tau$ (see Example 3.8 below). The condition $M_\tau = M_S$ is by [105, Proposition 3.7] equivalent to $\beta = \beta_e$, and by [105, Proposition 3.4], $(C_b(X), \beta_e)$ is a strong Mackey space. It is easy to find even locally compact X so that $(C_b(X), \beta)$ is not a Mackey space (for example, see [23, p. 481] or Example 3.9 below). A remarkable example of Haydon [44] has $\beta_0 = \beta$, $M_t = M_\tau \neq M_S$, and a result of [84, Theorem 3.2] says X admits a compatible complete uniform structure when $M_\tau = M_S$. Finally [84, Theorem 3.1], $M_S = M_\sigma$ iff X is a D-space in the sense of Granirer [38]; there are models for set theory in which every X is a D-space [38].

(e) Clearly (i) implies (ii), so assume (ii) holds and let the scalar valued function f on X have its restriction to each compact set be continuous. For each compact set λ, there exists $f_\lambda \in C_b(X)$ such that $\| f_\lambda \|_\infty \leqslant \| f \|_\infty$ and $f_\lambda = f$ on λ. The net $\{f_\lambda\}$, with the λ's ordered by inclusion, is norm (hence, β_0) bounded, so $E =$ the β_0 closure of the range of $\{f_\lambda\}$ is β_0 complete by (ii). Also, $\{f_\lambda\}$ is β_0 Cauchy, for, if $v \in V$ (Remark 3.3 above) and $\epsilon > 0$, and $\lambda_\epsilon = \{x \in X: v(x) \geqslant \epsilon/2 \| f \|_\infty\}$, then, $\lambda, \lambda' \geqslant \lambda_\epsilon$ implies $\| v(f_\lambda - f_{\lambda'})\|_\infty < \epsilon$. Thus, there is $g \in E$ such that $f_\lambda \to g(\beta_0)$, and it is clear that $g = f$. To conclude the proof of (e), assume (iii) holds and let $\{f_\alpha\}$ be a β_0 Cauchy net in $C_b(X)$. Since $\kappa \leqslant \beta_0$, $\{f_\alpha\}$ converges uniformly on compacta to an f that is continuous on compacta, so continuous by (iii). Suppose f is unbounded; then, there exists a sequence $\{x_n\}$ so that $| f(x_n)| \geqslant 2^{n+1}$, for all n. Let $v = \sum_{n=1}^{\infty} 2^{-n} f_n$, where f_n is the characteristic function of $\{x_n\}$. Then, $v \in V$ (3.3 above), so there is α_0 such that $\alpha \geqslant \alpha_0$ implies

$$| 2^{-n}[f_{\alpha_0}(x_n) - f_\alpha(x_n)]| \leqslant \| v(f_{\alpha_0} - f_\alpha)\|_\infty \leqslant 1,$$

and $| f_{\alpha_0}(x_n) - f(x_n)| = \lim_\alpha | f_{\alpha_0}(x_n) - f_\alpha(x_n)| \leqslant 2^n$, for all n. It follows that $| f_{\alpha_0}(x_n)| \geqslant 2^n$ for every n, which contradicts $f_{\alpha_0} \in C_b(X)$. Thus, $f \in C_b(X)$, and we need only show (since $\{f_\alpha\}$ is β_0 Cauchy) that f is a β_0 cluster point of $\{f_\alpha\}$. Let $v \in V$, $U = \{g \in C_b(X): \| gv \|_\infty \leqslant 1\}$, and α_0 be given. There exists $\alpha_1 \geqslant \alpha_0$ such that $\alpha \geqslant \alpha_1$ implies $f_{\alpha_1} - f_\alpha \in U$, hence, $\kappa - \lim\{f_{\alpha_1} - f_\alpha : \alpha \geqslant \alpha_1\} \in U$, or $f_{\alpha_1} \in f + U$ (note the fact that U is κ closed was used), concluding the proof of (e) and also the proof of the theorem.

THEOREM 3.5. (a) *$\beta_0 = \beta$ iff $M_t = M_\tau$ and X is Prohorov ($M_t = M_\tau$ is needed here)*;

(b) *$\beta_0 = \beta$ and $M_t = M_\tau$ when X is either locally compact (here $\beta_0 =$ Buck's original topology) or a complete metric space or a P-space; in the latter two cases $(C_b(X), \beta_0)$ is a strong Mackey space*;

(c) *if X is a hemicompact k'-space or the topological sum of a family of hemicompact k spaces, then, $(C_b(X), \beta_0)$ is a strong Mackey space and X is Prohorov.*

Proof. (a) See [82, Theorem 5.2, 44]; (b) if X is locally compact, 3.1 above and [82, Theorem 5.4] show $\beta_0 = \beta$ (hence, $M_t = M_\tau$, by Theorem 3.4(a) above). A direct proof using Summers' definition of β_0 (3.3 above) is more straightforward. One simply verifies directly that $\beta_0 =$ Buck's topology. To show that X is Prohorov, let $H \subset M_t^+$ be a weak* compact set. Since H is certainly uniformly norm bounded, we need only prove that H is uniformly tight, i.e., [85, Theorem 4.8], that $\| e_\lambda \cdot \mu - \mu \| \to 0$ uniformly for $\mu \in H$. Here, λ runs through all compacta, $e_\lambda \in C_c(X)$ is such that $0 \leqslant e_\lambda \leqslant 1$, $e_\lambda = 1$ on λ, and $\| e_\lambda \cdot \mu - \mu \| = \sup\{| \int \varphi(1 - e_\lambda)\, d\mu | : \varphi \in C_0(X), \| \varphi \|_\infty \leqslant 1\}$. Let $\epsilon > 0$ be given and note that the sets $\{\mu \in M_t(X): | \int (1 - e_\lambda)\, d\mu | < \epsilon\}$ form a weak* open cover of H. There exists $\lambda_1, \lambda_2, \ldots, \lambda_n$ so that $H \subset \bigcup_{i=1}^{\infty} \{\mu \in M_t(X): | \int (1 - e_{\lambda_i})\, d\mu | < \epsilon\}$. If $\lambda_0 = \bigcup_{i=1}^{n} \operatorname{spt} e_{\lambda_i}$ and $\lambda \geqslant \lambda_0$, then $\| e_\lambda \cdot \mu - \mu \| < \epsilon$, and this completes the proof that X is Prohorov.

Assume now X is a complete metric space and let $H \subset M_t^+$ be weak* compact, $\epsilon > 0$, and let $H_1 = \{\mu \in H: \mu(X) \geqslant \epsilon\}$, a weak* compact set. If $J = \{\mu/\| \mu \| : \mu \in H_1\}$, then J is a weak* compact subset of tight probability measures on X and so, by [9, p. 240], J is uniformly tight. But then, there is K compact in X so that $1/\| \mu \|\, \mu(X \backslash K) < \epsilon/\sup\{\| \mu \| : \mu \in H_1\}$, for all $\mu \in J$, hence, $\mu(X \backslash K) < \epsilon$, all $\mu \in H$. This proof is due to Sentilles. See [82, Theorem 9.2] for the proof that $\beta_0 = \beta$ and [33, Theorem 4] for the proof that $(C_b(X), \beta_0)$ is a strong Mackey space.

To finish (b), suppose X is a P-space. We simply sketch the proof here (see [106] for details). Wheeler shows first [106, Theorem 2.1] that $M_\tau = M_t$ and in [106, Theorem 2.2] shows $(C_b(X), \beta_0)$ is a strong Mackey space and $\beta_0 = \beta$. The l^∞ trick is used, and begins by supposing H is a subset of $M_t = M_\tau$ that is weak* relatively countably compact, but not uniformly tight. There then exists $\epsilon > 0$ and pairwise disjoint compact (hence, finite) subsets D_n and $\mu_n \in H$ such that $| \mu_n | (X \backslash \bigcup_{i=1}^{n-1} D_i) > \epsilon$ and $| \mu_n | (X \backslash \bigcup_{i=1}^{n} D_i) < \epsilon/4$, for every n. Since X is a P-space, the collection $\{D_n\}$ is discrete [36], and there exists a pairwise disjoint sequence $\{F_n\}$ of closed sets such that $D_n \subset \operatorname{int} F_n$ and $\{F_n\}$ is also discrete. If $D_n = \{x_{i,n} : 1 \leqslant i \leqslant i_n\}$, there is f_n in $C_b(X)$ such that $f_n(x_{i,n}) = \operatorname{sgn} \mu_n(\{x_{i,n}\})$, $f_n \mid (X \backslash F_n) \equiv 0$, and $\| f_n \|_\infty \leqslant 1$.

The map $T: l^\infty \to C_b(X)$ defined by $T(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_n$ is $\beta_0 - \beta_0$ continuous, so the adjoint $T': M_t \to l^1$ is weak* continuous and $T'(H)$ is weak* relatively countably compact, hence, relatively norm compact. There is then n_0 such that $|T'(\mu_n)(e_n)| < \epsilon/2$, $n \geqslant n_0$, where $e_n =$ the nth unit vector of l^1. However, $|T'(\mu_n)(e_n)| = |\mu_n(f_n)| > \epsilon/2$, a contradiction.

(c) See [64, pp. 884–885].

THEOREM 3.6. *Suppose X is as in the preceding. Then:*

(a) *X is measure compact iff $\beta = \beta_1$, in which case X is realcompact* [36] (*the converse is not true*).

(b) *For metrizable X, the following are equivalent*: (i) *X is measure compact*, (ii) *$(C_b(X), \beta_1)$ is complete*, (iii) *each closed discrete subspace of X is measure compact.*

(c) *$(C_b(X), \beta)$ is a strong Mackey space whenever X is measure compact, or metrizable, or paracompact* (*in the latter case, $\beta_e = \beta$*).

Proof. We simply give references except for a sketch of part of (c), which again uses the l^∞ trick. The proof of (a) is found in [82, Theorem 5.6, 33, Proposition 5, 51, Theorem 3.2]. A counter-example for (a) is in [42–44]. Part (b) of the theorem may be found in [33, Theorem 7]. The proof for (c) for X measure compact is in [82, Theorem 4.5], while the proof when X is paracompact was reduced by Wheeler [105, Proposition 3.8] to the case that X is metrizable. A sketch of the proof for X metrizable will now be given [105, Theorem 2.1]; this proof uses a basic technique developed in [33, Theorem 4], where it is shown $(C_b(X), T_t)$ is a strong Mackey space when X is a complete metric space (here $\beta_0 = \beta$). We assume d is a compatible metric for X and $H \subset M_t$ is relatively weak* countably compact. The l^∞ trick appears in proving the assertion: $\epsilon > 0$, $\delta > 0$ imply there is finite $Y \subset X$ such that for all $\mu \in H$, $|\mu|(X \backslash N(Y, \delta)) \leqslant \epsilon$, where $N(Y, \delta) = \{x \in X: d(x, Y) \leqslant \delta\}$. If this fails for some ϵ and δ, there exist sequences $\{\mu_n\}$ in H and $\{Y_n\}$ finite subsets of X such that (1) $Y_0 = \varnothing$, (2) $|\mu_n|(X \backslash N(U_{i<n} Y_i, \delta)) > \epsilon$, (3) $Y_n \subset X \backslash N(\bigcup_{i<n} Y_i, \delta)$, and (4) $|\mu_n|(N(Y_n, \delta/4)) > \epsilon$, all n. Let $G_n = \{x \in X: d(x, Y_n) < \delta/3\}$, $H_n = N(Y_n, \delta/4)$. Since $|\mu_n|(H_n) > \epsilon$, there is $f_n \in C_b(X)$ such that $\|f_n\|_\infty \leqslant 1$, $f_n \mid (X \backslash G_n) \equiv 0$, and $|\int f_n \, d\mu_n| > \epsilon$. Define $T: l^\infty \to C_b(X)$ by $T(\alpha) = \sum_{n=1}^{\infty} \alpha_n f_n$. Then, T is $\sigma(l^\infty, l^1) - \sigma(C_b, M_\tau)$ continuous, so $T': M_\tau \to l^1$ is weak*–weak* continuous, and $T'(H)$ is relatively weak* countably compact in l^1, hence, relatively norm compact. But if e_n = the nth unit vector of l^1, then $| T'(\mu_n)(e_n)| = | \mu_n(f_n)| > \epsilon$, a contradiction.

Hence, for each m and n there is a finite subset $F_{m,n}$ of X such that $| \mu | (X \backslash N(F_{m,n}, 1/2^n)) \leqslant (m \cdot 2^n)^{-1}$, all $\mu \in H$. If

$$S = \mathrm{cl} \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} N(F_{m,n}, 1/2^n),$$

then S is separable and $| \mu | (X \backslash S) = 0$, all $\mu \in H$. Let μ_S be the restriction of μ to S. If $\{\mu_n\}$ is a sequence in H with a weak* cluster point μ_0 in $M_\tau(X)$, then $| \mu_0 | (X \backslash S) = 0$ [103, p. 183]. It follows $\{\mu_{n_S}\}$ clusters to μ_{0_S} in $M_\tau(S)$, so $H_1 = \{\mu_S : \mu \in H\}$ is relatively weak* countably compact in $M_\tau(S)$. Since S is Lindelöf (hence, $M_\sigma(S) = M_\tau(S)$ [103, p. 175]), H_1 is $(C_b(S), \beta)$ equicontinuous. But then, H is $(C_b(X), \beta)$ equicontinuous.

THEOREM 3.7. (a) *X is strongly measure compact if X is σ compact.*

(b) *X is β simple when X is a hemicompact k' space.*

(c) *X strongly measure compact implies the following conditions are equivalent*: (i) *X is Prohorov,* (ii) *X is β simple,* (iii) *$(C_b(X), \beta_0)$ is a Mackey space,* (iv) *$(C_b(X), \beta_0)$ is a strong Mackey space.*

(d) *If X is a σ compact metric space, the following are equivalent*: (i) *$(C_b(X), \beta_0)$ is a Mackey space or a strong Mackey space,* (ii) *X is Prohorov,* (iii) *X has a compatible complete metric,* (iv) *X contains no G_δ subspace homeomorphic to the rationals.*

(e) *The following are equivalent*: (i) *$(C_b(X), \beta_1)$ is a weighted space,* (ii) *X is β simple,* (iii) *X is strongly measure compact and Prohorov, and* (iv) *(when X is also the topological sum of a family $\{X_\alpha : \alpha \in A\}$ of hemicompact k spaces) A, with its discrete topology, is measure compact.*

Proof. Part (a) is found in [64, Proposition 3.4], (b) in [64, Theorem 5.2], (c) in [64, Theorem 2.13], and (d) in [64, Theorem 5.8], where some deep work of Preiss [69] is needed. Part (e) uses our Theorem 2.4 above, together with [64, Proposition 2.13 and Corollary 5.5].

EXAMPLE 3.8. Let X be a non-Lebesgue measurable subset of [0, 1] that has outer Lebesgue measure one and contains the dyadic points. Let μ_n be the atomic measure giving mass 2^{-n} to each of the points $k/2^n$, $k = 1, 2, 3, \ldots$, and let $\nu_n = \mu_n - \mu_{n+1}$. Then, $\nu_n \to 0 \sigma(M_t, C_b)$, while $\{\mu_n\} = \{| \nu_{n-1} |\}$ converges $\sigma(M_t, C_b)$ to an element of $M_\tau \backslash M_t$.

$D = \{\mu_n\}$ is a relatively weak* compact set in M_τ but not in M_t. $C = \{\nu_n\}$ is relatively weak* compact in M_t, but C is *not* uniformly tight. The weak* closed absolutely convex hull E of C is weak* compact in M_t, but not uniformly tight. Thus, $(C_b(X), \beta_0)$ is not a Mackey space and $M_t \neq M_\tau$.

EXAMPLE 3.9. As remarked by Conway [23, p. 481], there is a class of spaces X where the l^∞ trick cannot be used to prove $(C_b(X), \beta_0)$ is a Mackey space, namely, the pseudocompact noncompact, but locally compact spaces. Such a space is the space X of ordinals less than the first uncountable ordinal, with the order topology. The relevant set H is the weak* closed absolutely convex hull of the set of measures $\frac{1}{2}[\delta_x - \delta_{x+1}]$, where $x \in X$, δ_x is the point mass at x, and x is a nonlimit ordinal. The set H is weak* compact, but not uniformly tight. Note also that it is observed in [20, p. 77] that $C_0(X)$ does not have an approximate identity whose range is $\sigma(C_b, M_t)$ relatively compact and thus, not one which is β totally bounded. The latter follows from [20, Theorem 3.10], for X is *not* paracompact.

A final comment in this section seems appropriate. McKinney [59], employing the kernel techniques developed by Sentilles in [79–81], has shown that for $\xi = \beta_1$ or β, $(C_b(X), \xi)$ is a Mackey space iff it is a strong Mackey space. Although here the method does not work for $\xi = \beta_0$ [56, pp. 396–397], it may work for $\xi = \beta_e$.

4. APPLICATIONS TO APPROXIMATION THEORY

In this section, we discuss briefly four areas of approximation theory in which strict or weighted topologies have played an important part, namely, (1) spectral analysis of bounded continuous functions, (2) analytic functions, (3) two norm and Saks spaces, and (4) weighted approximation.

Very little can be said about (1), although the early use of strict and weighted topologies was made here. We do include here several results along these lines; the proofs will be omitted except for the one of Theorem 4.4.

DEFINITION 4.1. Let G be an LCA (locally compact Abelian group) that is σ compact, and φ be a strictly positive function in $C_0(G)$. The *narrow topology* ν on $C_b(G)$ is the metric topology given by $d(f, g) =$

$\| \varphi(f - g)\|_\infty + \big| \|f\|_\infty - \|g\|_\infty \big|$, $f, g \in C_b(G)$. Note that $f_\alpha \to f(\nu)$ iff $f_\alpha \to f(\kappa)$ and $\|f_\alpha\|_\infty \to \|f\|_\infty$, and so one can define narrow convergence on non-σ compact groups. It is clear that narrow convergence is strictly stronger than β convergence.

Remark 4.2. As given above, ν is akin to a weighted topology, given by the single weight φ. It is not a weighted topology in the sense of 2.2 above, however, for ν is not even a *linear* topology. Narrow convergence was a presager of weighted spaces and was introduced very early by Beurling [5] for G the real line, where he proved the remarkable

THEOREM 4.3 (Beurling). *Let f be a bounded uniformly continuous function on $(-\infty, \infty)$ and let Λf be the ν closed linear span of the set of translates of f. Then, $\Lambda f \neq \varnothing$, and in fact, $e^{i\lambda x} \in \Lambda f$ for some real λ.*

The proof given by Beurling used complex function theory, but later, Koosis in [52] gave a different and more general proof (applicable to arbitrary LCA's). Herz obtained a similar result, using β instead of ν, and we now state and prove his (weaker)

THEOREM 4.4 (Herz). *Let G be an* LCA *and $f \neq 0$, $f \in C_b(G)$. Then, there exists a real λ such that $e^{i\lambda x}$ is in the β closed linear span in $C_b(G)$ of the translates of f.*

Proof. The key lemma here is (we consider $C_b(G)$ as a subset of $L^\infty(G)$) the result that the relative $\sigma(L^\infty, L^1)$ and β closures for translation invariant subspaces L of $C_b(G)$ coincide. Given this for the moment, as remarked by Koosis [52, p. 121], the usual Wiener–Tauberian theorem may be used to complete the argument. Koosis calls "Theorem On Spectral Analysis" the analog of 4.4 with $f \in L^\infty$, so (in view of the lemma), Herz's result is weaker than, yet similar to, both it and Beurling's theorem.

We now prove the lemma. By Buck's theorem that $(C_b(G), \beta)' = M_b(G)$, regarding $L^1(G)$ as a subset of $M_b(G)$, we need only show $f \in C_b(G)$ and $f \notin \beta$ closure E of L implies $f \notin \sigma(L^\infty, L^1)$ closure of L. Let $\{\varphi_\lambda\}$ be a bounded approximate identity of norm $\leqslant 1$ for $L^1(G)$ consisting of $\varphi_\lambda \in C_c(G)^+$. An easy argument verifies that $\varphi_\lambda * g \to g(\beta)$ for all $g \in C_b(G)$ and that E is a $C_c(G)^+$ *-module. Since $\varphi_\lambda * f \to f(\beta)$, there is $\varphi =$ some φ_λ such that $\varphi * f \notin \varphi * E$. By the Separation theorem and Buck's theorem, there exists $\mu \in M_b(G)$ such that $\langle \varphi * f, \mu\rangle \neq 0$, but $\langle \varphi * E, \mu\rangle = 0$. Let $\tilde{\varphi}(x) = \varphi(-x)$ and $\psi = \mu * \tilde{\varphi} \in L^1(G)$. Note that if $\langle f, \psi \rangle = \int f(x)\,\varphi(x)\,dx$, then $\langle f, \psi \rangle = \langle \varphi * f, \mu \rangle \neq 0$, while $\langle E, \psi \rangle = \langle \varphi * E, \mu \rangle = 0$. The proof is concluded.

In [37, p. 332], use is made of β to obtain a result in spectral synthesis.

In connection with (2), a basic theorem in the study of analytic functions in a domain (which uses the theory of normal families) is that each uniformly bounded net of analytic functions that converges pointwise (such a net is said to *converge boundedly*) also converges uniformly on compacta (and hence, converges β). This remark enables one to prove easily the following basic theorem, which inspired Rubel, Shields, and Ryff [75, 76, 78] to make an extensive study of the topological algebra $(H^\infty(G), \beta)$, the bounded analytic functions in the domain G, with the strict topology β. In the theorems that follow, it is assumed that $H^\infty(G)$ separates points of G. A proof of 4.5 may be found in [76].

THEOREM 4.5. *Let G be a domain in the complex plane and let $\{f_\alpha\}$ be a uniformly bounded net in $H^\infty(G)$, $f \in H^\infty(G)$. The following are then equivalent*:

(a) $f_\alpha \to f(\beta)$;
(b) $f_\alpha \to f$ *pointwise*; *and*
(c) $f_\alpha \to f(\kappa)$.

In particular, a sequence $f_n \to f$ boundedly iff either it converges β or is uniformly bounded and converges uniformly on compacta.

From the above theorem and Definition 4.1, (which applied here says $f_\alpha \to f(\nu)$ iff both $f_\alpha \to f(\kappa)$ and $\|f_\alpha\|_\infty \to \|f\|_\infty$), it is seen that the topology ν, introduced by Rubel in [75], is strictly finer than β.

Certain other topologies were introduced in $H^\infty(G)$, namely, the weak topology α, the bounded-weak* topology γ, the Mackey topology m, the strongest topology τ on $H^\infty(G)$ in which $f_n \to f(\tau)$ iff $f_n \to f$ boundedly, and the mixed topology $\mu = m(H^\infty, t^*, t)$, where $t^* = \kappa$ and t is the norm topology (recall 2.2 above). One of the motivations for these topologies is found in the following

THEOREM 4.6. *For $f \in H^\infty(G)$ and $\mu \in M_b(G)$, let $\langle f, [\mu] \rangle = \int f\,d\mu$, where $[\mu] \in M'(G) = M(G)/N(G)$, with the usual quotient norm and $N(G) =$ all $\lambda \in M_b(G)$ such that $\int f\,d\lambda = 0$ for all $f \in H^\infty(G)$. Then*:

(a) $M'(G) = (H^\infty(G), \beta)'$, α *is the weak topology in the above* *pairing, hence, α and β have the same duals $M'(G)$ and the same bounded sets (namely, the norm-bounded sets).*

(b) *$(H^\infty(G), \beta)$ is semireflexive and $H^\infty(G)$ is the conjugate of the separable Banach space $M'(G)$.*

(c) *$\alpha \neq \beta = \tau = \gamma = \mu$, and $\beta \neq m$, $\beta \neq \nu$, in general.*

Proof. Parts (a) and (b) are proved in [76], as is the fact that $\alpha \neq \beta$. The first proof that $\beta \neq m$ was given (for $G = D$, the open unit disc) by Conway in [22]. More general results of this type may be found in [78, p. 172], as well as the proofs (pp. 170, 172) that $\beta = \gamma$ and $\beta = \tau$.

THEOREM 4.7. (a) *A linear subspace of $H^\infty(G)$ is β closed iff it is β sequentially closed.*

(b) *The principal ideal (f) is dense in $(H^\infty(G), \beta)$ iff f is outer.*

(c) *(f) is β closed iff the outer factor of f is a unit.*

(d) *Every β closed ideal in $H^\infty(G)$ is the principal ideal generated by an inner function.*

Proof. See [76, 4.8, 5.1, 5.4, and 5.5].

Rubel remarks in [75, p. 18] that perhaps the main reason for introducing topologies on spaces of functions is for approximation problems, and an early result [29] says that any f defined on the bounded domain G is the bounded limit of a sequence of polynomials iff f has a bounded analytic extension to the outer boundary of G. Related results are discussed in [75, p. 18], where conditions are considered in which pointwise boundedly dense subsets E of $H^\infty(G)$ are strongly pointwise boundedly dense (which amounts to saying $f \in H^\infty(G)$ implies there is $\{f_n\}$ in E so that $f_n \to f(\beta)$ iff the same can be said with β replaced by ν). Sentilles in [80, 82] also gives (in a general setting) a fine discussion of sequentially defined topologies, and McKennon in [58] makes excellent use of what he calls the topology $\sigma\nu(E, B)$, where E is the conjugate of some Banach space B, in which $e_\lambda \to e(\sigma\nu)$ iff $\| e_\lambda \| \to \| e \|$ and $e_\lambda \to e$ pointwise on B (note that this, in case $B = M'(G)$ and $E = H^\infty(G)$ is the narrow convergence ν). Finally, some general theorems related to these results on $(H^\infty(G), \beta)$ may be found in [86–88].

We shall say very little about (3), not because it is unimportant, but simply because our interests in strict topologies lie elsewhere. However, it is clear that Wiweger's interest in his mixed topologies was motivated largely by the two norm and Saks spaces (see [107]), for there, the basic theory is built around the fact that $f_n \to f$ iff $\|f_n - f\|^* \to 0$ and $\sup \|f_n\| < \infty$ (here one has a linear space E with a pseudonorm $\| \ \|^*$ and a norm $\| \ \|$). With certain additional relations and restrictions assumed by Wiweger, this (what he calls) γ convergence becomes a mixed convergence. In case $E = C_b(X)$, with X a hemicompact space, $\|f\|^* = \sum_{n=1}^{\infty} 2^{-n} \operatorname{Artcan} \|f\|_n$, $\{K_n\}$ is a base for compacta in X, $\|f\|_n = \sup\{|f(x)| : x \in K_n\}$, and $\|f\| = \sup\{|f(x)| : x \in X\}$, then $m(E, \| \ \|^*, \| \ \|) = \beta_0$ and $\beta_0 = \beta_0{}^+ =$ the finest LCS topology on E for which $f_n \to f$ iff $f_n \to f(\gamma)$ (see [107, Theorems 2.6.1 and 3.1, 80, p. 533]). A similar convergence is discussed in [40, pp. 166–168, 170].

The final application in this section (where *weighted* topologies play the primary role) involves the weighted approximation problem for modules, which extends the classical Bernstein approximation problem in much the same sense that the Stone–Weierstrass theorem contains the classical Weierstrass theorem. (An excellent survey of much of this theory may be found in [65, 93].) The first theorem below indicates somewhat the interesting role of $(C_b(X), \beta_0)$ as a predictor of results to be expected in the weighted spaces (E_V, ω_V), where V is a Nachbin family on X, and ω_V is assumed to be Hausdorff (see 2.2 above). The results we list are the most important ones with respect to their contributions to a solution of the weighted approximation problem.

THEOREM 4.8. (a) *If V dominates $U =$ all characteristic functions of compacta and X is a k' space, then (E_V, ω_V) is complete*;

(b) *if X is locally compact and $C(X)^+$ dominates V, then $(E_V, \omega_V)' = V \cdot M_b(X)$, the extreme points of $V_v{}^0$ are given by $\{\lambda v(x)\, \delta_x : x \in \operatorname{coz} v, |\lambda| = 1\}$, each $v \in V$, and $V_v{}^0$ is the polar of $V_v = \{f \in E_V : \|vf\|_\infty \leq 1\}$ in $(E_V, \omega_V)'$ and $V_v{}^0 = v \cdot B$, where B is the closed unit ball in $M_b(X)$.*

Proof. For (a), see [89, Corollary 3.7] and for (b), see [90, Theorems 3.1, 4.5, 4.6].

The result in [18, Theorem 3.1] that $(C_b(X), \beta_0)$ has (when X is locally compact) the approximation property of Grothendieck has been extended to the general case in [33, Theorem 10] and to the spaces (E_V, ω_V) in [6, 7]. A particularly important result, namely Bishop's generalized Stone–Weierstrass theorem [37], has been extended and generalized in many directions. We state now some of these results; a definition is needed first.

DEFINITION 4.9. A subset K of $C_b(X)$ is *antisymmetric* for a subset B of $C_b(X)$ if $f \in B$ implies $f \mid K$ is constant whenever $f \mid K$ is real valued.

THEOREM 4.10 (Bishop–Glicksberg). *Suppose X is locally compact, A is closed subspace of $(C_b(X), \beta)$, and B is a subset of $C_b(X)$ (which we may, without loss of generality, assume to be a β closed subalgebra) such that $BA \subseteqq A$. Then:*

(a) *Every antisymmetric for B is contained in a maximal such, and the collection $\mathscr{K}_B$ of maximal antisymmetric sets for B forms a closed pairwise disjoint cover of X.*

(b) *If $f \in C_b(X)$ and $f \mid K$ is in the β closure of $A \mid K$ for every $K \in \mathscr{K}_B$, then $f \in A$.*

Proof. See [37, p. 330].

As is pointed out by Glicksberg [37, p. 332], the above Theorem 4.10 yields a nice generalized Stone–Weierstrass theorem for $(C_b(X), \beta)$, namely,

THEOREM 4.11. *Let X be locally compact, A a β closed selfadjoint subalgebra of $C_b(X)$ that separates points of X. Then, $A = C_b(X)$.*

Proof. In Theorem 4.10, take $B = A$ and note that any antisymmetric set for A is a singleton. Now, $A \mid \{x\} =$ all constants, so $A = C_b(X)$.

Before proceeding to a discussion of the weighted approximation problem, we need several definitions.

DEFINITION 4.12. Let X be a completely regular T_2 space, let A be an algebra of continuous complex valued functions on X that contains the constants, let V be a Nachbin family on X (see 2.2 above), and let W be a linear subspace of E_V that is also an A-module.

(a) The *weighted approximation problem* [65, 66] asks for a description of the closure cl W in (E_V, ω_V); in particular, by a *localization* of the weighted approximation problem relative to a pairwise disjoint closed cover $\mathscr{K}$ of X, we mean that $f \in$ cl W iff $f \mid K$ belongs to the closure of $W \mid K$ in $(E_{V\mid K}, \omega_{V\mid K})$, for every $K \in \mathscr{K}$.

(b) W is *localizable under* A (in the sense of Nachbin) when the localization is possible relative to $\mathscr{K}_1$, where $\mathscr{K}_1 =$ all $\sim$ equivalence classes, with $x \sim y$ iff $a(x) = a(y)$, all $a \in A$.

(c) The *bounded case* of the weighted approximation problem occurs when each $a \in A$ is bounded on the support of each $v \in V$.

(d) W is *A-localizable whenever* cl W in (E_V, ω_V) is precisely

$\{f \in E_V : K \in \mathscr{K}_A,\ v \in V$, and $\epsilon > 0$ implies there is $w \in W$ such that $\sup\{v(x) \mid f(x) - w(x) \mid : x \in K\} < \epsilon\}$. Here, $\mathscr{K}_A$ is the collection of maximal antisymmetric sets for A.

With these definitions, we now can state the major results on weighted approximation.

THEOREM 4.13. (a) *If A is selfadjoint, then W is A-localizable iff W is localizable under A and hence, W is A-localizable in the bounded selfadjoint case.*

(b) *W is always A-localizable in $(C_b(X), \beta_0)$ in the bounded case.*

(c) *If X is locally compact and V is a Nachbin family dominated by $C(X)^+$, then W is A-localizable in the bounded case.*

(d) *In general, W is A-localizable in the bounded case (even if A is not selfadjoint).*

Proof. The first part of (a) follows from the definitions and the second part was proved by Nachbin in [66, p. 295].

Part (b) is Summers' result for the space $(E_V, \omega_V) = (C_b(X), \beta_0)$, with V as in 3.3 above, and a proof is found in [92, Theorem 3.1]. The proof of part (c) is due to Prolla and is found in [70, p. 284]. (d) The proof for this best possible result, at least for the bounded case of the weighted approximation problem, is due to Summers [95, Theorem 2.1].

Remark 4.14. Note that the results (b)–(d) above include Theorem 4.10. To see this, suppose X is locally compact, A is a β closed subspace of $C_b(X)$, and B is a β closed subalgebra of $C_b(X)$ so that A is a B-module. Since X is locally compact, $\beta_0 = \beta$ by 3.5 above. Applying (b), we know A is B-localizable in $(C_b(X), \beta)$, and this is precisely the statement (b) of 4.10. To see that (c) implies 4.10, let $V = C_0(X)^+$, so that $(E_V, \omega_V) = (C_b(X), \beta)$.

Theorem 4.13 (d) is an extremely general result of the Stone–Weierstrass type that subsumes many noteworthy results. For example, since $(C_b(X), \beta_0)$ is a weighted space, recent approximation results [33, p. 134; 35, p. 470; 92, p. 97] are corollaries. Finally, Summers (to appear) has succeeded, just as Nachbin did for the selfadjoint case, in using his theorem to establish a more general criterion of A-localizability called the *analytic case* [65, p. 91]. This is the situation where there exist subsets $G(A)$ of A and $G(W)$ of W such that (i) the subalgebra of A generated by $G(A)$ is compact-open dense in A, (ii) the A-submodule of W generated by $G(W)$ is ω_V dense in W, and (iii) $v \in V$, $a \in G(A)$, $w \in G(W)$ implies there exists $\alpha > 0$, $\beta > 0$ such that $|w(x)| v(x) \leqslant \alpha e^{-\beta |a(x)|}$ for all $x \in X$. Note that these three conditions clearly hold for A and W in the bounded case, and Summers' result is: *W is always A-localizable in the analytic case of the weighted approximation problem.*

5. Some Noncommutative Results

As expected, this section is devoted to generalizations to not necessarily commutative settings of previous theorems on strict, weighted, and mixed topologies on function spaces. The first known result of this type (with applications to extensions of C^*-algebras) is Busby's [15], where he notes that if $M(A)$ is the double centralizer algebra of the C^*-algebra A and (we regard A as an ideal of $M(A)$) β is given (for $x \in M(A)$) by the seminorms $x \to \max(\| ax \|, \| xa \|)$, $a \in A$, then, $(M(A), \beta)$ is a complete locally convex topological algebra containing A as a β dense ideal, and $M(A)$ is the largest subalgebra of A'' (the bidual of A under Arens multiplication) containing A as a two sided ideal. Note that we can require a to be in $V =$ the positive elements of A, and this is a cone in A and so might be regarded as a Nachbin family of weights (when A is Abelian, so that $A = C_0(S)$, then these weights are continuous and vanish at ∞). The fact that $M(A) = \{x \in A'' \colon xV \cup Vx \subseteq A\}$ is the above one that $M(A)$ is the idealizer of A in A''. Thus, one sees that $(M(A), \beta)$ is a "weighted" space. As remarked earlier, Buck [13] shows $(M(A), \beta) = (C_b(S), \beta)$ when $A = C_0(S)$. From another point of view, $M(A)$ is the (Grothendieck completion of $(A, \beta \mid A)$ [15, Proposition 3.5 and 3.6, or 57], and a comprehensive study of various completions of more general A and their relation to the regularity of Arens multiplication on A'' may be found in [57].

A somewhat more general setting, at least in some respects, is found in [19, 85, 87, 88], where one studies a "left strict" topology on a left Banach A-module E (here, A is a Banach algebra with bounded approximate identity). For $x \in E$, $x \to \| ax \|$, $a \in A$, gives a defining family for β, and so β is akin to a weighted topology, and generalizes the case $(C_b(S), \beta)$ by taking $A = C_0(S)$ and $E = C_b(S)$. Some interesting new techniques are developed and used in [85] (based upon the Cohen–Hewitt factorization theorem), and almost all of Buck's basic results about β are generalized, including

THEOREM 5.1. *Let E be a left Banach A-module, with β the left strict topology. Then*:

(a) $(E, \beta)' = E' \cdot A$.

(b) *The β equicontinuous subsets of $(E, \beta)'$ are those norm bounded sets H in $E' \cdot A$ such that $\| x' \cdot e_\lambda - x' \| \to 0$ uniformly for $x' \in H(x' \cdot a(x) = x'(ax))$, where $\{e_\lambda\}$ is a bounded approximate identity for A).*

(c) *When $A = C_0(S)$ and $M(A) = E = C_b(S)$, then $E' \cdot A = M_b(S) = M_t(S)$ and in this case,* (b) *is exactly Conway's result that H is β equicontinuous iff it is norm bounded and uniformly tight.*

(d) *The left strict topology β is the strict topology in our sense* [80, *Theorem* 2.2], *i.e.*, $\beta = \beta'$.

In [Theorem 5.1], Sentilles obtains in this general setting of a left Banach A-module E a sufficient condition for (E, β) to be a Mackey space (which also uses the l^∞ trick and yields the Le Cam–Conway result when $A = C_0(S)$, $E = C_b(S)$, and S is σ-compact). Shapiro, motivated by the above, shows in [87] that every linear subspace of E having β compact unit ball is a conjugate Banach space whose bounded weak* topology $= \beta$, an extension to left Banach modules of some of the work in [17, 86].

The rest of this section reverts (except for some remarks at the end on $\Gamma(K)$) back to "two sided" strict topologies on $M(A)$, A a C^*-algebra, as defined by Busby. The first efforts here are those of Taylor [98], where he extends many of the results previously known for the Abelian case $A = C_0(S)$.

THEOREM 5.2. *Let A be a C^*-algebra, with $M(A)$ its double centralizer algebra, β the strict topology on $M(A)$ defined by A (see the first part of* 5), *and let $\{e_\lambda\}$ be a positive bounded approximate identity of norm $\leqslant 1$. Then*:

(a) $(M(A), \beta)' = M(A)' \cdot A = A \cdot M(A)' \simeq A' \cdot A = A \cdot A'$.

(b) *$H \subset (M(A), \beta)'$ is β equicontinuous iff H is norm bounded and $e_\lambda \cdot x' + x' \cdot e_\lambda - e_\lambda \cdot x' \cdot e_\lambda \to x'$ uniformly for $x' \in H$.*

(c) *$\beta = \beta'$, so β is strict in the sense of Definition* 2.1.

(d) *$(M(A), \beta)'$ is a Mackey space (resp. strong Mackey space) iff A is the subdirect sum* [71] *of C^*-algebras A_δ such that $(M(A_\delta), \beta_\delta)$ is a Mackey space (resp. a strong Mackey space).*

(e) $(M(A), \beta)$ *is a strong Mackey space if either A has a countable approximate identity, or $A = C_0(S)$ and S is locally compact paracompact, or $M(A) = A''$.*

(f) $M(A)' = (M(A), \beta)' \oplus A^{\perp}$, *where $A^{\perp}$ is the annihilator of A in $M(A)'$.*

Proof. All the above (except (f)) are found in [98, Theorems 2.1, 2.6, Corollaries 2.3, 2.7, 3.3]. The proof of (f) is in [100, Corollary 2.7]. It is of interest to note that the l^{∞} trick is used in the proof of (e).

EXAMPLE 5.3. if $A = C_0(S)$, with S locally compact, it is known [18, Theorem 4.1] that A has a countable approximate identity iff S is σ compact. Thus (d) and (e) above together generalize the Conway–Le Cam result to this non-Abelian setting.

EXAMPLE 5.4. Let A be the C^*-algebra of compact operators in the Hilbert space H, in which case $M(A)$ is both A'' and *all* bounded operators in H. In fact, $M(A) = A''$ if A is a subdirect sum of A_δ's with each A_δ the compact operators in a Hilbert space H_δ [57].

The importance of approximate identities to a study of strict topologies can be clearly seen in some of the above theorems. It becomes even more apparent in the following theorems, where not only the existence, but the *type* of approximate identities is crucial. For the sake of simplicity, we again restrict ourselves to the C^*-algebra setting.

DEFINITION 5.5. *Let A be a C^*-algebra with an approximate identity* $\{e_\lambda : \lambda \in \Lambda\}$ such that $e_\lambda \geqslant 0$ and $\| e_\lambda \| \leqslant 1$. Then,

(a) $\{e_\lambda\}$ is *canonical* [20] provided $\lambda_1 < \lambda_2$ implies $e_{\lambda_1} e_{\lambda_2} = e_{\lambda_1}$ (this generalizes to this non-Abelian case the notion introduced in [18, p. 158]).

(b) $\{e_\lambda\}$ is *well behaved* if in addition to being canonical, it also satisfies the condition: $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, $\lambda \in \Lambda$ implies there is N such that $m, n > N$ implies $e_\lambda e_{\lambda_m} = e_\lambda e_{\lambda_n}$.

(c) $\{e_\lambda\}$ is β *totally bounded* provided the range of $\{e_\lambda\}$ is β totally bounded.

THEOREM 5.6. (a) *If A is the subdirect sum of C^*-algebras each with a well-behaved approximate identity, then A has one.*

(b) *If A has either a countable or series* [1, p. 527] *approximate* *identity, or if* $A = C_0(S)$ *with* S *paracompact, then* A *has a well-behaved approximate identity.*

(c) *If* A *has a well-behaved approximate identity, then*: (i) *if* $\{f_n\}$ *is a sequence in* $M(A)'$ *that converges weak* to zero, then* $\{f_n{}^0\}$ *is uniformly tight and converges weak* to zero (note each* $f \in M(A)'$ *is uniquely representable as* $f = f^0 + f^1$, *with* $f^0 \in (M(A), \beta)'$ *and* $f^1 \in A^\perp$; *see* 5.2 (f) *above*); (ii) $(M(A), \beta)$ *is a strong Mackey space*; (iii) $(M(A), \beta)'$ *is weak* sequentially complete*; *and* (iv) *if* X *is a Banach space and* T *is a bounded linear map from* X *to* $M(A)$ *such that* $T(X) + A = M(A)$, *then there is* λ *such that* $(1 - e_\lambda)\, M(A)(1 - e_\lambda) = (1 - e_\lambda)\, T(X)(1 - e_\lambda)$.

Proof and Remarks. All of the proofs are found in [99], as well as a discussion of how these results relate to others. We will sketch only one of them, namely, (c, i). This result generalizes from the commutative case a result of Conway [24, Theorem 2.2], who used the l^∞ trick.

Proof of Theorem 5.6 (c, i). It is clear that we may assume $\| f_n{}^0 \| \leqslant 1$ for all n. If $\{f_n{}^0\}$ is not uniformly tight (i.e., not β-equicontinuous), there is $\epsilon > 0$ so that $\sup_n \|(1 - e_\lambda) f_n{}^0(1 - e_\lambda)\| \geqslant 4\epsilon$, for all λ. We can then define sequences $n_1 < n_2 < n_3 < \cdots$ and $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ such that $\|(1 - e_{\lambda_k}) f^0_{n_k}(1 - e_{\lambda_k})\| \geqslant 4\epsilon$ and $\| e_{\lambda_{k+1}} f^0_{n_k} e_{\lambda_{k+1}} - f^0_{n_k} \| < \epsilon$. It then follows that

$$\|(1 - e_{\lambda_k})\, e_{\lambda_{k+1}} f^0_{n_k} e_{\lambda_{k+1}} (1 - e_{\lambda_k})\| = \|(e_{\lambda_{k+1}} - e_{\lambda_k}) f_{n_k} (e_{\lambda_{k+1}} - e_{\lambda_k})\| \geqslant 3\epsilon.$$

For each k choose $b_k = b_k{}^*$ in the unit ball of A such that

$$\| f_{n_k}[(e_{\lambda_{k+1}} - e_{\lambda_k})\, b_k (e_{\lambda_{k+1}} - e_{\lambda_k})]\| \geqslant \epsilon.$$

Define $a_k = (e_{\lambda_{2k+1}} - e\lambda_{2k})\, b_{2k}(e_{\lambda_{2k+1}} - e_{\lambda_{2k}})$, and let $g_k = f_{n_{2k}}$. Then we have (1) $| g_k(a_k)| \geqslant \epsilon$, (2) $a_j a_k = 0$ if $j \neq k$, (3) for each λ there exists N such that $a_k e_\lambda = 0$ for $k \geqslant N$. Let $\alpha = \{\alpha_k\} \in l^\infty$, and note by (2) and (3) that $\{\sum_{k=1}^n \alpha_k a_k\}$ is uniformly bounded and β Cauchy and hence, converges β to $T(\alpha) = \sum_{k=1}^\infty \alpha_k a_k$, since $(M(A), \beta)$ is complete [15, p. 83]. T is a bounded operator from l^∞ to $M(A)$, so $T'\colon M(A)' \to (l^\infty)'$ exists and $\{T'(g_k)\}$ converges weak* to zero. A basic result of Phillips says $\lim_{m\to\infty} \sum_{q=m}^\infty | T'(g_k)(\delta_q)| = \lim_{m\to\infty} \sum_{q=m}^\infty | g_k(a_k)| = 0$ uniformly in k, where δ_k is the Kronecker delta function. Thus, there is m such that $| g_m(a_m)| \leqslant \sum_{q=m}^\infty | g_m(a_q)| < \epsilon$, and this contradicts (1), so $\{f_n{}^0\}$ is uniformly tight. To complete (c, i), note $\{f_n{}^0\}$ is β equicontinuous and converges pointwise on the β dense subset A of $M(A)$. Thus, $\{f_n^0\}$ converges weak* to zero, concluding the theorem.

Some final informal remarks concerning other results now complete this section. Phillips' theorem that c_0 is not complemented in l^∞ follows readily from Taylor's work [99, Corollary 3.7] or Conway's [24, Theorem 3.5], as does a fine result of Bade on interpolation [2, Theorem 1.1]. In [100], Taylor (using the strict topology) characterizes the conjugate space of a maximal full algebra of operator fields and develops a noncommutative analog of the theory of interpolation for function algebras. In [20], an extensive study is made of various kinds of approximate identities on C^*-algebras. In particular [20, Theorem 3.10], it is shown that $A = C_0(S)$ has a β totally bounded approximate identity iff S is paracompact, and examples are given that seem to indicate that no clear connection exists between β totally bounded and well-behaved approximate identities. Fontenot in [30–32] has extended many results in topological measure theory to the present context of $(M(A), \beta)$, including proper generalizations of results about tight, τ-additive and σ-additive functionals on $M(A)$. In a series of pepers [53, 54 and others submitted], Lazar and Taylor develop and study a noncommutative analog of the topological algebra of complex valued continuous functions on a locally compact space S, with the compact-open topology κ, denoted by them $(\Gamma(K), \kappa)$. Here K is Pedersen's ideal of the C^*-algebra A, κ is the kappa topology with seminorms $(x \in \Gamma(K))\ x \to \max(\| kx \|, \| xk \|)$, $k \in K$, and $\Gamma(K)$ is the double centralizer algebra of K. Their work includes (aside from the expected generalizations from the Abelian case) a spectral theory and functional calculus for $\Gamma(K)$, a study of derivations of $\Gamma(K)$, and a Dauns–Hoffman theorem for $\Gamma(K)$.

This author is attempting to develop a noncommutative theory of weighted spaces, based upon $\Gamma(K)$ as the analog of the continuous scalar valued functions. For example, it is fairly easy to see how such a theory might start: We let V be a "Nachbin family" in $\Gamma(K)^+$ (= the positive cone of $\Gamma(K)$) and E_V = all $x \in \Gamma(K)$ such that $xV \cup Vx \subseteq A$, ω_V the topology whose seminorms are $(x \in E_V)\ x \to \max(\| yx \|, \| xy \|)$, $y \in V$. If $V = K^+$, then $(E_V, \omega_V) = (\Gamma(K), \kappa)$; if $V = A^+$, $(E_V, \omega_V) = (M(A), \beta)$, and if V is the set of nonnegative constants (= the cone generated by the identity of $\Gamma(K)$), then $(E_V, \omega_V) = (A, \| \ \|)$. Each of these spaces (E_V, ω_V) is a Hausdorff topological algebra that generalizes from the Abelian case $A = C_0(S)$ (resp.) the spaces $(C(S), \kappa)$, $(C_b(S), \beta)$, and $C_0(S)$. Note in these examples that $V \subseteq \Gamma(K)^+$ and so consists of "continuous" weights. To emulate the commutative theory of weighted spaces of functions one wants to allow u.s.c. not necessarily continuous nor bounded objects. Recently, Pedersen [68] has defined (in the noncommutative setting) the "real" lower semicontinuous elements of A'' as the set A^m of operators in A'' that can be approached weak* from below by selfadjoint operators of the form $x + \alpha$, with $x \in A$ and α real. Then, $A_m = -A^m$ is the set of bounded u.s.c. operators and $A^m \cap A_m =$ the selfadjoint part of $M(A)$ (Theorem 2.5). Thus, a method exists for studying weighted spaces E_V in $\Gamma(K)$ arising from Nachbin families of bounded u.s.c. operators. However, at least for now, no reasonable method seems available for lifting the boundedness restriction on the members of V, so that V would be simply a Nachbin family of u.s.c. "operators" and E_V would be all $x \in \Gamma(K)$ so that xv and vx are bounded and vanish at ∞, for all $v \in V$.

6. Conclusion

The preceding account of these topologies and their applications is quite sketchy, and many important results (particularly those of the Buchwalter school) are either omitted or touched upon only briefly. Credit may not always be given to the proper person (through our ignorance). Another very serious defect inherent in a short survey paper is the fact that many important techniques and proofs are left out, for such proofs and techniques often yield more information than the theorems themselves. The set theory and cardinality problems intrinsic in discussions of measure compactness and realcompactness were simply avoided in our survey, and various results on vector-valued measures that utilize strict topologies [6–7, 8, 13, 14, 44, 104, etc.] were not emphasized. Again, our definition of strict topology may well be open to criticism, as being too general and nonfunctional. It is hoped that an expanded version of this paper (in preparation) will correct most of these deficiences.

References

1. C. A. Akemann, Interpolation in W^*-algebras, *Duke Math. J.* **35** (1968), 525–534.
2. W. G. Bade, Extensions of interpolating sets, functional analysis, in "Proceedings of a Conference held at the University of California, Irvine," (B. R. Gelbaum, Ed.), Thompson, Washington, 1967.
3. J. Berruyer and B. Ivol, L'espace $M(T)$, *C.R. Acad. Sci. Paris* **275** (1972), 33–36.

4. J. BERRUYER AND B. IVOL, Une topologie sur l'espace des mesures de Riesz, *C. R. Acad. Sci. Paris* **274** (1972), 1927–1930.
5. A. BEURLING, Un théorème sur les fonctions bornées et uniformément continues sur l'axe reel, *Acta Math. (Uppsala)* **77** (1945), 127–136.
6. K. BIERSTEDT, Gewichtete Räume stetiger vektorwertiger Funktionen und das injektive Tensorprodukt I, *J. Reine Angew. Math.* **259** (1973), 186–210.
7. K. BIERSTEDT, Gewichtete Räume stetiger vektorwertiger Functionen und das injektive Tensoproduct II, *J. Reine. Angew. Math.* **260** (1973), 133–146.
8. K. BIERSTEDT AND R. MEISE, Lokalkonvexe Unterräume in topologischen Vektorräumen und das ϵ-Produkt, *Manuscripta Math.* **8** (1973), 143–172.
9. P. BILLINGSLEY, "Convergence of probability measures," Wiley, New York, 1968.
10. E. BISHOP, A generalization of the Stone–Weierstrass theorem, *Pac. J. Math.* **11** (1961), 777–783.
11. R. C. BUCK, Operator algebras and dual spaces, *Proc. Amer. Math. Soc.* **3** (1952), 681–687.
12. R. C. BUCK, Algebraic properties of classes of analytic functions, Seminars on analytic functions II, p. 175–188, Institute for advanced study, Princeton, N. J., 1957.
13. R. C. BUCK, Bounded continuous functions on a locally compact space, *Mich. J. Math.* **5** (1958), 95–104.
14. H. BUCHWALTER, "Lecture Notes." Vol. 331, pp. 183–202. Springer–Verlag, New York, 0000.
15. R. C. BUSBY, Double centralizers and extensions of C^*-algebras, *Trans. Amer. Math. Soc.* **132** (1968), 79–99.
16. H. S. COLLINS, Completeness and compactness in linear topological spaces, *Trans. Amer. Math. Soc.* **79** (1955), 256–280.
17. H. S. COLLINS, On the space $l^\infty(S)$, with the strict topology, *Math. Z.* **106** (1968), 361–373.
18. H. S. COLLINS AND J. R. DORROH, Remarks on certain function spaces, *Math. Ann.* **176** (1968), 157–168.
19. H. S. COLLINS AND W. H. SUMMERS, Some applications of Hewitt's factorization theorem, *Proc. Amer. Math. Soc.* **21** (1969), 727–733.
20. H. S. COLLINS AND R. A. FONTENOT, Approximate identities and the strict topology, *Pacific J. Math.* **43** (1972), 63–79.
21. J. B. CONWAY, The strict topology and compactness in the space of measures, *Bull. Amer. Math. Soc.* **72** (1966), 75–78.
22. J. B. CONWAY, Subspaces of $(C_b(S), \beta)$, the space (l^∞, β), and (H^∞, β), *Bull. Amer. Math. Soc.* **72** (1966), 79–81.
23. J. B. CONWAY, The strict topology and compactness in the space of measures II, *Trans. Amer. Math. Soc.* **126** (1967), 474–486.
24. J. B. CONWAY, A theorem on sequential convergence of measures and some applications, *Pacific J. Math.* **28** (1969), 53–60.
25. J. B. COOPER, The strict topology and spaces with mixed topology, *Proc. Amer. Math. Soc.* **30** (1971), 583–592.
26. J. R. DORROH, The localization of the strict topology via bounded sets, *Proc. Amer. Math. Soc.* **20** (1969), 413–414.
27. Y. DOMAR, On spectral analysis in the narrow topology, *Math. Scand.* **3–4** (1955–56), 328–332.
28. R. M. DUDLEY, Convergence of Baire measures, *Studia Math.* **27** (1966), 251–268.

29. O. J. Farrell, On approximation by polynomials to a function analytic in a simply connected region, *Bull. Amer. Math. Soc.* **41** (1934), 707–711.
30. R. A. Fontenot, Approximate identities and the strict topology, Dissertation, Louisiana State University, 1972.
31. R. A. Fontenot, Strict topologies for vector valued functions, to appear.
32. R. A. Fontenot, Topological vector space properties of double centralizer algebras, to appear.
33. D. H. Fremlin, D. J. H. Garling, and R. G. Haydon, Bounded measures on topological spaces, *Proc. Lond. Math. Soc.* **25** (1972), 115–136.
34. D. J. H. Garling, A generalized form of inductive limit topology for vector spaces, *Proc. Lond. Math. Soc.* **14** (1964), 1–28.
35. Robin Giles, A generalization of the strict topology, *Trans. Amer. Math. Soc.* **161** (1971), 467–474.
36. L. Gillman and M. Jerison, "Rings of Continuous Functions," Van Nostrand, Princeton, N.J., 1960.
37. I. Glicksberg, Bishop's generalized Stone–Weierstrass theorem for the strict topology, *Proc. Amer. Math. Soc.* **14** (1963), 329–333.
38. E. Granirer, On Baire measures on D-topological spaces, *Fund. Math.* **60** (1967), 1–22.
39. A. Grothendieck, Sur les espaces (F) et (DF), *Summa. Bras. Math.* **3** (1954), 57–122.
40. D. Gulick, The σ-compact-open topology and its relatives, *Math. Scand.* **30** (1972), 159–176.
41. D. Gulick and J. Schmets, Separability and seminorm separability for spaces bounded continuous functions, *Bull. Soc. Roy. Sci. Liège*, **41** (1972), 254–260.
42. Richard Haydon, Sur les espaces $M(T)$ et $M^{\infty}(T)$, *C. R. Acad. Sci. Paris* **275** (1972), 989–991.
43. Richard Haydon, Sur un problème de H. Buchwalter, *C. R. Acad. Sci. Paris* **275** (1972), 1077–1080.
44. Richard Haydon, On compactness in spaces of measures and measure compact spaces, Preprint.
45. C. S. Herz, The spectral theory of bounded functions, *Trans. Amer. Math. Soc.* **94** (1960), 181–232.
46. J. Hoffmann Jørgensen, A generalization of the strict topology, *Math. Scand.* **30** (1972), 313–323.
47. B. E. Johnson, An introduction to the theory of centralizers, *Proc. Lond. Math. Soc.* **14** (1964), 299–320.
48. B. E. Johnson, Centralizers on certain topological algebras, *J. London Math. Soc.* **39** (1964), 603–614.
49. R. B. Kirk, Measures in topological spaces and B-compactness, *Nederl. Akad. Wetensch. Proc. Ser. A* **72** (1969), 172–183.
50. R. B. Kirk, Locally compact, B-compact spaces, *Nederl. Akad. Wetensch. Proc. Ser. A* **72** (1969), 333–344.
51. J. D. Knowles, Measures on topological spaces, *Proc. Lond. Math. Soc.* **17** (1967), 139–156.
52. Paul Koosis, On the spectral analysis of bounded functions, *Pacific J. Math.* **16** (1966), 121–128.
53. A. Lazar and D. C. Taylor, Double centralizers of Pedersen's ideal of a C^*-algebra, *Bull. Amer. Math. Soc.* **78** (1972), 992–997.

54. A. LAZAR AND D. C. TAYLOR, Double centralizers of Pedersen's ideal of a C^*-algebra II, *Bull. Amer. Math. Soc.* **79** (1973), 361–366.
55. L. LE CAM, Convergence in distribution of stochastic processes, *Statistics* **2** (1957), 207–236.
56. C. LEGER AND P. SOURY, Le convexe topologique des probabilitées sur un espace topologique, *J. Math. Pures Appl.* **50** (1971), 363–425.
57. E. MCCHAREN, Arens multiplications on locally convex completions of Banach algebras, Dissertation, Louisiana State University, 1970.
58. KELLY MCKENNON, *Mem. Amer. Math. Soc.* **3** (1971).
59. JUDITH MCKINNEY, Kernels of measures on completely regular spaces, *Duke Math. J.* **40** (1973), 915–923.
60. W. MORAN, The additivity of measures on completely regular spaces, *J. Lond. Math. Soc.* **43** (1968), 633–639.
61. W. MORAN, Measures and mappings on topological spaces, *Proc. Lond. Math. Soc.* **19** (1969), 493–508.
62. W. MORAN, Measures on metacompact spaces, *Proc. Lond. Math. Soc.* **20** (1970), 507–524.
63. S. E. MOSIMAN, Strict topologies and the ordered vector space $C_b(X)$, unpublished manuscript.
64. S. E. MOSIMAN AND R. F. WHEELER, The strict topology in a completely regular setting: relations to topological measure theory, *Canad. J. Math.* **24** (1972), 873–890.
65. L. NACHBIN, "Elements of Approximation Theory," Van Nostrand, Princeton, N. J., 1967.
66. L. NACHBIN, Weighted approximation for algebras and modules of continuous functions: real and self-adjoint complex cases, *Ann. Math.* **81** (1965), 289–302.
67. L. NACHBIN, S. MACHADO, AND J. PROLLA, Weighted approximation, vector fibrations, and algebras of operators, *J. Math. Pures Appl.* **50** (1971), 299–323.
68. GERT PEDERSEN, Applications of weak* semicontinuity in C^*-algebra theory, *Duke Math. J.*, to appear.
69. D. PREISS, Metric spaces in which Prohorov's theorem is not valid, *Z. Wahrsch. Verw. Gebiete* **27** (1973), 109–116.
70. J. B. PROLLA, Bishop's generalized Stone–Weierstrass theorem for weighted spaces, *Math. Ann.* **191** (1971), 283–289.
71. C. RICKART, General theory of Banach algebras, Van Nostrand, Princeton, N. J., 1960.
72. MICHEL ROME, Le dual de l'espace compactologique $C^\infty(T)$, *C. R. Acad. Sci. Paris* **274** (1972), 1631–1634.
73. MICHEL ROME, Ordre et compacite dans l'espace $M^\infty(T)$, *C. R. Acad. Sci. Paris* **274** (1972), 1817–1820.
74. A. P. ROBERTSON AND W. ROBERTSON, "Topological Vector Spaces," Cambridge, Univ. Press, New York, 1964.
75. L. A. RUBEL, Bounded convergence of analytic functions, *Bull. Amer. Math. Soc.* **77** (1971), 13–24.
76. L. A. RUBEL AND A. L. SHIELDS, The space of bounded analytic functions on a region, *Ann. Inst. Fourier Grenoble*, **16** (1966), 235–277.
77. L. A. RUBEL AND A. L. SHIELDS, The second duals of certain spaces of analytic functions, *J. Austr. Math. Soc.* **11** (1970), 276–280.

78. L. A. Rubel and J. V. Ryff, The bounded weak-star topology and the bounded analytic functions, *J. Functional Analysis* **5** (1970), 167–183.
79. F. Dennis Sentilles, Compactness and convergence in the space of measures, *Ill. J. Math.* **13** (1969), 761–768.
80. F. Dennis Sentilles, The strict topology on bounded sets, *Pacific J. Math.* **34** (1970), 529–540.
81. F. Dennis Sentilles, Compact and weakly compact operators on $(C_b(S), \beta)$, *Ill. J. Math.* **13** (1969), 769–776.
82. F. Dennis Sentilles, Bounded continuous functions on a completely regular space, *Trans. Amer. Math. Soc.* **168** (1972), 311–336.
83. F. Dennis Sentilles, Conditions for equality of the Mackey and strict topologies, *Bull. Amer. Math. Soc.* **76** (1970), 107–112.
84. F. Dennis Sentilles and R. F. Wheeler, Linear functionals and partitions of unity in $C_b(X)$, to appear.
85. F. Dennis Sentilles and D. C. Taylor, Factorization in Banach algebras and the general strict topology, *Trans. Amer. Math. Soc.* **142** (1969), 141–152.
86. Joel H. Shapiro, Weak topologies on subspaces of $C_b(S)$, *Trans. Amer. Math. Soc.* **157** (1971), 471–479.
87. Joel H. Shapiro, The bounded weak star topology and the general strict topology, *J. Functional Analysis* **8** (1971), 275–286.
88. Joel H. Shapiro, Noncoincidense of the strict and strong operator topologies, *Proc. Amer. Math. Soc.* **35** (1972), 81–87.
89. W. H. Summers, A representation theorem for biequicontinuous completed tensor products of weighted spaces, *Trans. Amer. Math. Soc.* **146** (1969), 121–131.
90. W. H. Summers, Dual spaces of weighted spaces, *Trans. Amer. Math. Soc.* **151** (1970), 323–333.
91. W. H. Summers, Factorization in Fréchet spaces, *Studia Math.* **39** (1971), 209–216.
92. W. H. Summers, The general complex bounded case of the strict weighted approximation problem, *Math. Ann.* **192** (1971), 90–98.
93. W. H. Summers, Weighted spaces and weighted approximation, Séminaire d'analyse moderne, Sherbrooke, 1970.
94. W. H. Summers, Weighted approximation for modules of continuous functions, *Bull. Amer. Math. Soc.* **79** (1973), 386–388.
95. W. H. Summers, Weighted approximation for modules of continuous functions II, to appear.
96. W. H. Summers, The bounded case of the weighted approximation problem, to appear.
97. W. H. Summers, Separability in the strict and substrict topologies, *Proc. Amer. Math. Soc.* **35** (1972), 507–514.
98. D. C. Taylor, The strict topology for double centralizer algebras, *Trans. Amer. Math. Soc.* **150** (1970), 633–643.
99. D. C. Taylor, A general Phillips theorem for C^*-algebras and some applications, *Pacific J. Math.* **40** (1972), 477–488.
100. D. C. Taylor, Interpolation in algebras of operator fields, *J. Functional Analysis* **10** (1972), 159–190.
101. C. Todd, Stone–Weierstrass theorem for the strict topology, *Proc. Amer. Math. Soc.* **16** (1965), 654–659.
102. A. C. M. Van Rooij, Tight functionals and the strict topology, *Kyungpook Math. J.* **7** (1967), 41–43.

103. V. S. VARADARAJAN, Measures on topological spaces, *Mat. USSR-Sb*, **97** (1961), 35–100; *Amer. Math. Soc. Transl.* **48** (1965), 161–228.
104. J. WELLS, Bounded continuous vector valued functions on a locally compact space, *Michigan Math. J.* **11** (1965), 119–126.
105. R. F. WHEELER, The strict topology, separable measures, and paracompactness, *Pacific J. Math.* **47** (1973), 287–302.
106. R. F. WHEELER, The strict topology for *P*-spaces, *Proc. Amer. Math. Soc.* **41** (1973), 466–472.
107. A. WIWEGER, Linear spaces with mixed topology, *Studia Math.* **20** (1961), 47–68.